

# 6. ISOMETRIES

## §6.1. Isometries

Fundamental to the theory of symmetry are the concepts of distance and angle. So we work within  $\mathbb{R}^n$ , considered as an inner-product space. This is the usual  $n$ -dimensional real vector space, together with the inner product defined by the dot product  $\mathbf{u} \cdot \mathbf{v}$ . We use this dot product to define distances and angles.

Recall that the length of a vector  $\mathbf{v}$  is given by  $|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$  and that the distance between two points  $\mathbf{u}, \mathbf{v}$  is  $|\mathbf{u} - \mathbf{v}|$ . Moreover you will remember that the cosine of the angle  $\theta$  between two vectors  $\mathbf{u}, \mathbf{v}$  is given by  $\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| \cdot |\mathbf{v}|}$ .

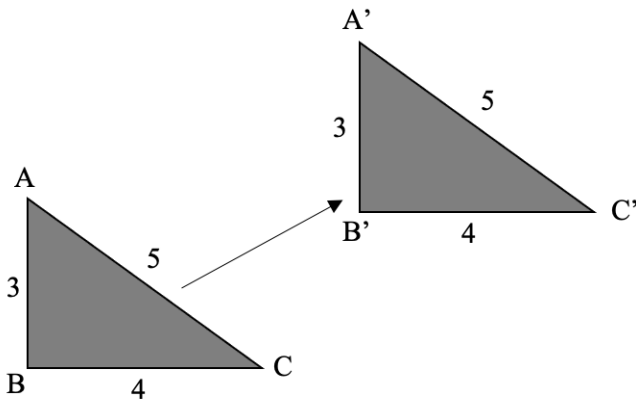
These correspond to the usual concept of distances and angles in 3-dimensional space and can be taken to be the definitions of lengths and angles in higher dimensional spaces (or in any inner product space for that matter).

An **isometry** is a distance-preserving function from  $\mathbb{R}^3$  to  $\mathbb{R}^3$ , that is, it is a function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that  $|f(\mathbf{u}) - f(\mathbf{v})| = |\mathbf{u} - \mathbf{v}|$  for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ . Examples include reflections, rotations and translations.

Now a fundamental tool in the study of isometries is linear algebra. However isometries need not be linear transformations since they needn't fix the origin – for

example translations don't. But those isometries that fix the origin *do* turn out to be linear, as we shall prove.

A **central isometry** is an isometry that fixes the origin. A reflection is a central isometry if the axis of reflection passes through the origin and a rotation is a central isometry if the centre of the rotation is the origin. A translation can never be central because it moves every point. However translations provide the link between a general isometry and a central one.



A **translation** is a function from  $\mathbb{R}^3$  to  $\mathbb{R}^3$  of the form  $f(\mathbf{v}) = \mathbf{v} + \mathbf{a}$  for some fixed  $\mathbf{a} \in \mathbb{R}^3$ . (If  $\mathbf{a} = \mathbf{0}$  the isometry is simply the identity map.)

**Theorem 1:** Every isometry is a central isometry or a central isometry followed by a translation.

**Proof:** Suppose  $f$  is an isometry. Define  $c(\mathbf{v}) = f(\mathbf{v}) - f(\mathbf{0})$ . This is an isometry followed by a translation and

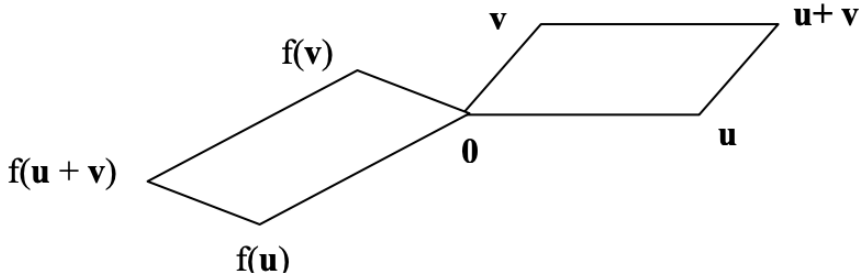
since the product of two isometries is an isometry,  $c$  is also an isometry. But  $c(\mathbf{0}) = \mathbf{0}$ , so  $c$  is a central isometry. Thus  $f(\mathbf{v}) = c(\mathbf{v}) + \mathbf{f}(\mathbf{0})$  is a central isometry followed by a translation.

## §6.2. Central Isometries

**Theorem 2:** Central isometries are linear transformations.

**Proof:** Suppose  $f$  is a central isometry. Isometries map straight lines to straight lines and so they map quadrilaterals to quadrilaterals. Since a parallelogram is a quadrilateral with opposite sides equal in length, isometries map parallelograms to parallelograms. A central isometry thus maps the parallelogram with vertices  $\mathbf{0}$ ,  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{u} + \mathbf{v}$  to the parallelogram with vertices  $\mathbf{0}$ ,  $f(\mathbf{u})$ ,  $f(\mathbf{v})$  and  $f(\mathbf{u}) + f(\mathbf{v})$ .

Thus  $f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$  for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ .



Suppose  $\lambda \neq 0$  and  $\mathbf{v} \neq \mathbf{0}$ . Clearly isometries take straight lines to straight lines and so, since  $\mathbf{0}$ ,  $\mathbf{v}$ ,  $\lambda\mathbf{v}$  are collinear, so are  $f(\mathbf{0}) = \mathbf{0}$ ,  $f(\mathbf{v})$  and  $f(\lambda\mathbf{v})$ .

Hence  $f(\lambda\mathbf{v})$  must be a scalar multiple of  $f(\mathbf{v})$ .

Suppose  $f(\lambda\mathbf{v}) = \mu f(\mathbf{v})$ .

Clearly, if  $\mathbf{v} = \mathbf{0}$  or  $\lambda = 0$  then  $f(\lambda\mathbf{v}) = \lambda f(\mathbf{v})$ .

So suppose that  $\lambda \neq 0$  and  $\mathbf{v} \neq \mathbf{0}$ .

Now  $|f(\lambda\mathbf{v})| = |\mu| \cdot |f(\mathbf{v})| = |\mu| \cdot |\mathbf{v}|$ .

But  $|f(\lambda\mathbf{v})| = |\lambda\mathbf{v}| = |\lambda| \cdot |\mathbf{v}|$ .

Hence  $\mu = \pm\lambda$  and so  $f(\lambda\mathbf{v}) = \pm\lambda f(\mathbf{v})$ .

However  $f(\lambda\mathbf{v}) = -\lambda f(\mathbf{v})$  then  $f((1 + \lambda)\mathbf{v}) = f(\mathbf{v}) + f(\lambda\mathbf{v})$   
$$= f(\mathbf{v}) - \lambda f(\mathbf{v})$$
$$= (1 - \lambda)f(\mathbf{v}),$$

and, taking lengths,  $(1 + \lambda)|\mathbf{v}| = (1 - \lambda)|f(\mathbf{v})| = (1 - \lambda)|\mathbf{v}|$ , a contradiction.

Thus  $f(\lambda\mathbf{v}) = \lambda f(\mathbf{v})$  for all  $\lambda$  and  $\mathbf{v}$ .

Hence  $f$  is a linear transformation.

Recall that an **orthogonal** matrix is a square real matrix  $A$  such that  $A^T A = I$ . Recall too that a set of vectors is an **orthonormal set** if they are mutually orthogonal unit vectors. By considering the product  $A^T A$  and comparing this with the identity matrix we see that the columns of an orthogonal matrix form an orthonormal set. It also follows from the equation  $A^T A = I$  that an orthogonal matrix  $A$  is invertible and that  $A^T = A^{-1}$ .

Now matrices don't commute in general, but every matrix commutes with its inverse, so we must also have  $AA^T = I$ . Writing this as  $(A^T)^T A^T$  we can see that if  $A$  is orthogonal then so is  $A^T$ . But the columns of  $A^T$  are the rows of  $A$  so the rows of an orthogonal matrix also form an orthonormal set.

Taking determinants of both sides of the equation  $A^T A = I$ , and using the fact that  $|A^T| = |A|$ , we can see that the determinant of an orthogonal matrix  $A$  must satisfy  $|A|^2 = 1$  and so must be  $\pm 1$ .

**Example 1:**  $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$  is an orthogonal matrix. The

columns

$\begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$ ,  $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$ ,  $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$  form an orthonormal set.

**Theorem 3:** The linear transformation  $f(\mathbf{v}) = A\mathbf{v}$ , on  $\mathbb{R}^3$  is an isometry if and only if  $A$  is orthogonal.

**Proof:** Suppose that  $f$  is an isometry. The standard basis vectors  $\mathbf{e}_1 = (1, 0, 0)$ ,  $\mathbf{e}_2 = (0, 1, 0)$ ,  $\mathbf{e}_3 = (0, 0, 1)$  are an orthonormal set. The columns of  $A$  are their images under  $f$  and so also form an orthonormal set. Thus  $A$  is orthogonal.

Conversely suppose that  $A$  is orthogonal and let  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ . Let  $\mathbf{w} = \mathbf{u} - \mathbf{v}$ .

Then  $|\mathbf{Aw}|^2 = (\mathbf{Aw})^T(\mathbf{Aw}) = \mathbf{w}^T A^T A \mathbf{w} = \mathbf{w}^T \mathbf{w} = |\mathbf{w}|^2$  so  $|\mathbf{Aw}| = |\mathbf{w}|$ .

Now  $|\mathbf{Au} - \mathbf{Av}| = |\mathbf{Aw}| = |\mathbf{w}| = |\mathbf{u} - \mathbf{v}|$  so  $f$  is an isometry.

Since the determinant of the matrix of a central isometry is  $\pm 1$  we can divide central isometries into two types. A central isometry is **direct** if it has determinant 1. It is **opposite** if it has determinant  $-1$ .

We can extend these ideas to isometries in general. An isometry is **direct** if it is a direct central isometry followed by a translation and **opposite** if it is an opposite central isometry followed by a translation.

The identity isometry is direct, and so are translations. To determine whether rotations and reflections in 2 and 3 dimensions are direct or opposite we need to find their matrices. But first let us consider isometries in 1 dimension.

We consider the very simple case of isometries of a line. The only  $1 \times 1$  real matrices with determinant  $\pm 1$

are the matrices (1) and  $(-1)$  and so the only central isometries of  $\mathbb{R}$  are the identity function  $I(x) = x$  and the reflection in the origin  $M(x) = -x$ .

Translation through a distance  $a$  is the isometry  $T_a(x) = x + a$ . The only other possible isometries are  $MT_a$  (for various values of  $a$ ). This is a reflection in the origin followed by a translation through  $a$ . If we let  $M_a = MT_a$  we get  $M_a(x) = T_a(M(x)) = T_a(-x) = a - x$ . This is a reflection in the point  $\frac{1}{2}a$ . Clearly these are all opposite isometries.

So the isometries of a line can be summarized as follows:

### LINEAR ISOMETRIES

ISOMETRY	Equation	DIRECT?	Fixed Points
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Identity	$I(x) = x$	$\checkmark$	all
Translation by $a$	$T_a(x) = x + a$	$\checkmark$	none
Reflection in $\frac{1}{2}a$	$M_a(x) = a - x$	$\times$	$a/2$

## §6.3. Central Isometries of the Plane

We shall show that the only central isometries of  $\mathbb{R}^2$  are rotations about the origin and reflections in a line that passes through the origin. Let  $\rho_\theta$  denote the **rotation** about the origin through the angle  $\theta$ .

Let  $\mu_\theta$  denote reflection in the line, through the origin, that is inclined at the angle  $\theta$  to the positive  $x$ -axis.

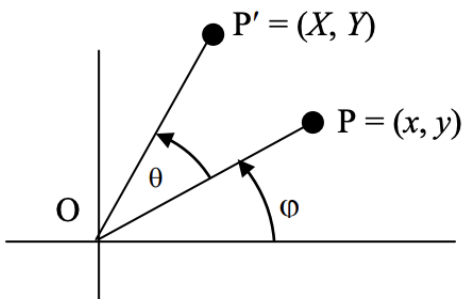
Each of these is a linear transformations of  $\mathbb{R}^2$  and so will correspond to a certain matrix. We'll find these matrices relative to the standard basis.



**Theorem 4:** The matrix for  $\rho_\theta$  is

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

**Proof:** Let  $P$  be the point  $(x, y)$  where  $x = r \cos\phi$  and  $y = r \sin\phi$ .



Rotating it about the origin through the angle  $\theta$  takes it to  $(X, Y)$  where:

$$X = r \cos(\phi + \theta) \text{ and } Y = r \sin(\phi + \theta).$$

We can write these equations as:

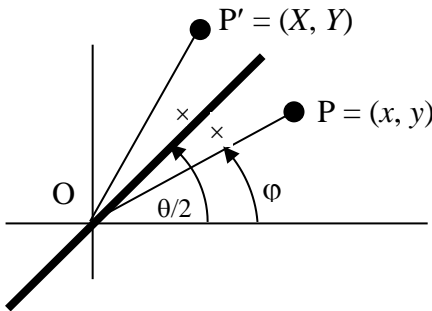
$$X = r \cos\phi \cdot \cos\theta - r \sin\phi \cdot \sin\theta = x \cos\theta - y \sin\theta \text{ and}$$

$$Y = r \sin\phi \cdot \cos\theta + r \cos\phi \cdot \sin\theta = x \sin\theta + y \cos\theta.$$

Writing this in matrix form we get:  $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

**Theorem 5:** The matrix for  $\mu_{\theta/2}$  is  $M_{\theta} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ .

**Proof:** Let  $x = r \cos\varphi$   
and  $y = r \sin\varphi$ .



The angles represented by  $\times$  are each equal to  $\theta/2 - \varphi$ .  
Hence  $\angle P'OX = \theta/2 + (\theta/2 - \varphi) = \theta - \varphi$  and so:

$$\begin{aligned} X &= r \cos(\theta - \varphi) \text{ and} \\ Y &= r \sin(\theta - \varphi). \end{aligned}$$

Expanding we get:

$$\begin{aligned} X &= r \cos(\theta - \varphi) \\ &= \cos\theta.r \cos\varphi + \sin\theta.r \sin\varphi \\ &= x \cos\theta + y \sin\theta \text{ and} \\ Y &= r \sin(\theta - \varphi) \\ &= \sin\theta.r \cos\varphi - \cos\theta.r \sin\varphi \\ &= x \sin\theta - y \cos\theta. \end{aligned}$$

In matrix form this becomes:  $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ .

Since  $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$ , rotations are direct isometries and

since  $\begin{vmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{vmatrix} = -1$ , reflections are opposite isometries.

**Theorem 6:** Every central isometry of the plane is either a rotation about the origin or a reflection in a line through the origin.

**Proof:** Let  $f$  be a central isometry of  $\mathbb{R}^2$ .

Write the corresponding orthogonal matrix as:  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Because of the orthogonality we have:

$$a^2 + c^2 = 1,$$

$$b^2 + d^2 = 1,$$

$$ab + cd = 0.$$

From the first equation we can write:

$a = \cos\theta$ ,  $c = \sin\theta$  for some  $\theta$ .

From the second we have:

$b = \cos\varphi$ ,  $d = \sin\varphi$  for some  $\varphi$ .

From the third equation we have:

$$\cos\theta \cdot \cos\varphi + \sin\theta \cdot \sin\varphi = 0.$$

Thus  $\cos(\theta - \varphi) = 0$ .

So  $\varphi = \pi/2 + \theta$ , in which case  $b = -\sin\theta$ ,  $d = \cos\theta$ ,  
 or  $\varphi = 3\pi/2 + \theta$ , in which case  $b = \sin\theta$ ,  $d = -\cos\theta$ .

Thus  $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$  or  $\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ .

The eigenvalues of the first matrix are  $e^{\pm i\theta}$  and of the second are  $\pm 1$ .

So the central isometries are:

ISOMETRY	Matrix	Eigenvalues
Identity	I	1, 1
Rotation through $\theta$ about the origin	$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$	$e^{i\theta}, e^{-i\theta}$
Reflection in the line $y = \tan(\theta/2).x$	$M_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$	1, -1

## §6.4. Products of Central Isometries

We can identify the product of some central isometries by multiplying the relevant matrices. But we must remember that the product of matrices must be in reverse order since the one that's applied first is the right hand factor.

**Example 2:** Identify the product,  $f$ , of a reflection in the line  $y = x$ , a rotation through  $60^\circ$  and a reflection in the  $y$ -axis.

**Solution:**

The matrices of each of these central isometries are, respectively (angles are in degrees):

$$\begin{pmatrix} \cos 90 & \sin 90 \\ \sin 90 & -\cos 90 \end{pmatrix}, \begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix}, \begin{pmatrix} \cos 180 & \sin 180 \\ \sin 180 & -\cos 180 \end{pmatrix}.$$

So we calculate  $M_{90}R_{90}M_{180}$

$$\begin{aligned} &= \begin{pmatrix} \cos 180 & \sin 180 \\ \sin 180 & -\cos 180 \end{pmatrix} \begin{pmatrix} \cos 60 & -\sin 60 \\ \sin 60 & \cos 60 \end{pmatrix} \begin{pmatrix} \cos 90 & \sin 90 \\ \sin 90 & -\cos 90 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \end{aligned}$$

So the product is a rotation through  $30^\circ$  about the origin.

**Theorem 7:**

$$R_\alpha R_\beta = R_{\alpha+\beta};$$

$$R_\alpha M_\beta = M_{\alpha+\beta};$$

$$M_\alpha R_\beta = M_{\alpha-\beta};$$

$$M_\alpha M_\beta = R_{\alpha-\beta}.$$

**Proof:** Just multiply the relevant matrices.

These results can be used to simplify the calculation of the product of central isometries.

**Example 3:** Identify the product,  $f$ , of a reflection in the line  $y = x$ , a rotation through  $60^\circ$  and a reflection in the  $y$ -axis.

**Solution:** The matrices of these isometries are  $M_{90}$ ,  $R_{60}$ ,  $M_{180}$ . We must multiply these in reverse order since the rightmost matrix is the one that corresponds to the first isometry to be applied.

$$M_{180}R_{60}M_{90} = M_{120}M_{90} = R_{30}.$$

The effect of the above identities is the application of the following rules.

(1) In a product of isometry matrices  $R$ 's and  $M$ 's the product is a **rotation** if there's an **even** number of reflections in the product and a **reflection** if there's an **odd** number of reflections.

(2) If the successive subscripts are  $\theta_1, \theta_2, \dots, \theta_n$  the subscript for the product is:  $\theta_1 \pm \theta_2 \pm \dots \pm \theta_n$  where each sign is  $+$  if there's an *even* number of  $M$ 's preceding the factor and  $-$  if there's an *odd* number.

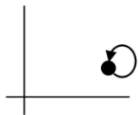
**Example 4:**  $M_\alpha R_\beta M_\gamma M_\delta R_\varepsilon = M_{\alpha-\beta-\gamma+\delta+\varepsilon}$

**Example 5:**  $R_\alpha M_\beta M_\gamma R_\delta R_\varepsilon = R_{\alpha+\beta-\gamma+\delta+\varepsilon}$

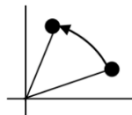
**Example 6:**  $M_{120}M_{45}M_{90}R_{60}M_{180} = R_{120-45+90-60-180}$   
 $= R_{-75} = R_{285}$ , a rotation about the origin through  $285^\circ$ .

## §6.5. Plane Isometries

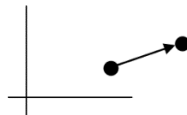
The central isometries of the plane are the identity,  $I$ , the rotations  $\rho_\theta$  and reflections in a line through the origin  $\mu_\theta$ . To complete the picture we need to consider these followed by translations.



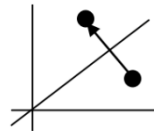
identity



rotation



translation



reflection

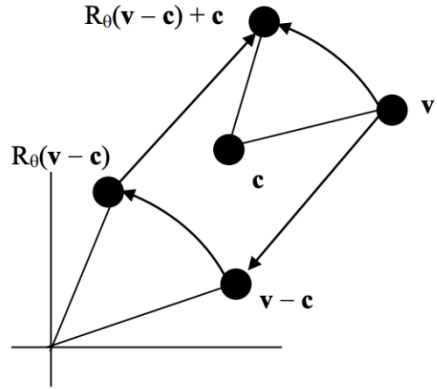
A **translation** is an isometry of the form  $\tau_{\mathbf{a}}$  where  $\tau_{\mathbf{a}}(\mathbf{v}) = \mathbf{v} + \mathbf{a}$ . This is said to be the translation by the vector  $\mathbf{a}$ .

We now need to follow rotations and reflections by translations. Perhaps a rotation followed by a translation is a rotation about some point other than the origin. If so, it will fix some point, and that will be the centre of the rotation. Let's see if we can solve the equation  $\rho_\theta \tau_{\mathbf{a}}(\mathbf{v}) = \mathbf{v}$  or, in other words,  $R_\theta \mathbf{v} + \mathbf{a} = \mathbf{v}$ . This gives  $(I - R_\theta)\mathbf{v} = \mathbf{a}$ . If  $I - R_\theta$  is invertible we can then write  $\mathbf{v} = (I - R_\theta)^{-1}\mathbf{a}$ .

Now, if  $R_\theta$  is a *proper* rotation, that is not through  $0^\circ$ , its eigenvalues are non-real. Hence 1 is not an eigenvalue and so  $|I - R_\theta| \neq 0$ . Thus it is indeed the case that  $I - R_\theta$  is invertible.

We've established that  $\rho_\theta \tau_{\mathbf{a}}$  fixes  $\mathbf{c} = (I - R_\theta)^{-1} \mathbf{a}$  but we have yet to establish that it's a rotation about the point  $\mathbf{c}$ .

Rotating the point  $\mathbf{v}$  about  $\mathbf{c} = (I - R_\theta)^{-1} \mathbf{a}$  through the angle  $\theta$  produces the point  $R_\theta(\mathbf{v} - \mathbf{c}) + \mathbf{c}$ . (We first make  $\mathbf{c}$  the temporary new origin, by subtracting  $\mathbf{c}$ , then rotate, and finally add  $\mathbf{c}$  to revert to the original origin.)



$$\begin{aligned} \text{Now } \rho_\theta \tau_{\mathbf{a}}(\mathbf{v}) &= R_\theta \mathbf{v} + \mathbf{a} \\ &= R_\theta(\mathbf{v} - \mathbf{c}) + \\ &R_\theta \mathbf{c} + \mathbf{a} \end{aligned}$$

$$\begin{aligned} &= R_\theta(\mathbf{v} - \mathbf{c}) + R_\theta(I - R_\theta)^{-1} \mathbf{a} + \mathbf{a} \\ &= R_\theta(\mathbf{v} - \mathbf{c}) + (R_\theta - I)(I - R_\theta)^{-1} \mathbf{a} + (I - R_\theta)^{-1} \mathbf{a} + \mathbf{a} \\ &= R_\theta(\mathbf{v} - \mathbf{c}) - \mathbf{a} + \mathbf{c} + \mathbf{a} \\ &= R_\theta(\mathbf{v} - \mathbf{c}) + \mathbf{c} \end{aligned}$$

Hence the product of this rotation and translation is a rotation through  $\theta$  about a new centre  $\mathbf{c} = (I - R_\theta)^{-1} \mathbf{a}$ .

Finally we must consider a central reflection followed by a translation. It would be natural to guess that this will be a reflection in some other line. However there's a surprise!

If the product is a reflection it will fix all the points on the axis of the reflection.

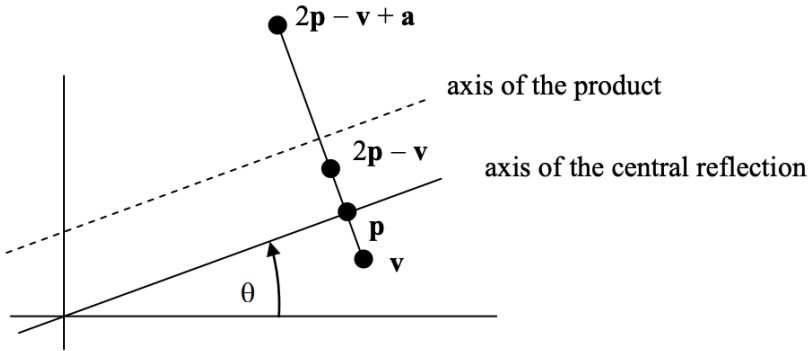
So let's consider the equation  $\mu_{\theta/2}\tau_{\mathbf{a}}(\mathbf{v}) = M_{\theta}\mathbf{v} + \mathbf{a} = \mathbf{v}$  or, equivalently,  $(I - M_{\theta})\mathbf{v} = \mathbf{a}$ .

This would give  $\mathbf{v} = (I - M_{\theta})^{-1}\mathbf{a}$  if  $I - M_{\theta}$  was invertible. But alarm bells should be ringing, for if that was the case then  $\mathbf{v}$  would be unique whereas we're expecting a whole line of fixed points.

The trouble is that  $I - M_{\theta}$  is not invertible. Remember that the eigenvalues of a reflection matrix are  $\pm 1$ . This makes sense because a central reflection fixes two lines – the axis of reflection, which is the eigenspace corresponding to the eigenvalue 1 and the perpendicular to that axis, which corresponds to the eigenvalue  $-1$ . Since 1 is an eigenvalue,  $|I - M_{\theta}| = 0$  and so  $I - M_{\theta}$  is not invertible.

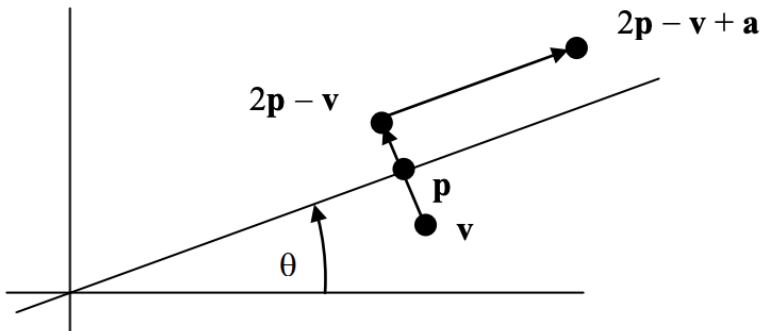
Let's first consider the special case where the direction of translation is *perpendicular* to the axis of reflection. In this case each point  $\mathbf{v}$  will stay on the line through  $\mathbf{v}$  perpendicular to the axis of reflection. The situation will become equivalent to the 1-dimensional case, which turned out to be a reflection.

Let  $\mathbf{p}$  be the foot of the perpendicular to the axis of the reflection. Then following the reflection  $\mathbf{v}$  will be sent to  $2\mathbf{p} - \mathbf{v}$ . (We can see that this is correct because the midpoint of  $\mathbf{v}$  and  $2\mathbf{p} - \mathbf{v}$  is  $\mathbf{p}$ , as it should be.) Translating



by  $\mathbf{a}$  will take the point to  $2\mathbf{p} - \mathbf{v} + \mathbf{a}$ . The midpoint of  $\mathbf{v}$  and  $2\mathbf{p} - \mathbf{v} + \mathbf{a}$  is  $2\mathbf{p} + \mathbf{a}$ . Since this is independent of  $\mathbf{v}$  each point will be mapped to its mirror image in the line through  $2\mathbf{p} + \mathbf{a}$ .

Now let's consider the special case where the direction of translation is *parallel* to the axis of reflection.

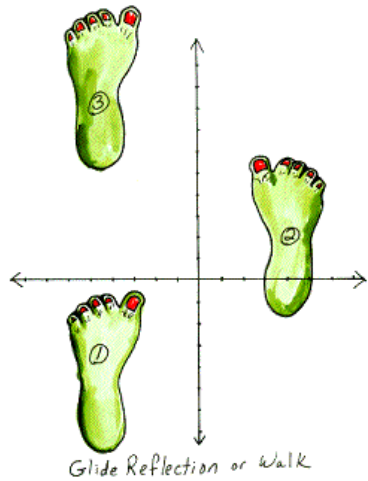


It's clear that in this case, assuming of course that  $\mathbf{a} \neq \mathbf{0}$ , the product fixes no points. Those that are not on the axis go to the other side, and those that are on the axis are fixed by the reflection and are then moved by the translation. So if there are no fixed points this composite can't be a reflection. Nor can it be a rotation or the identity. It must be a totally new type of isometry.

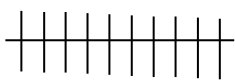
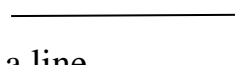
A **glide** is a reflection in a line followed by a translation parallel to that line.

It remains to consider a central reflection followed by a general translation, not necessarily in the direction perpendicular or parallel to the axis of reflection. In this case we simply resolve the vector that represents the translation in the parallel and perpendicular directions. That is, we write  $\mathbf{a} = \mathbf{b} + \mathbf{c}$  where  $\mathbf{b}$  is perpendicular to the axis and  $\mathbf{c}$  is parallel to it.

Then  $\mu_{\theta/2}\tau_{\mathbf{a}}(\mathbf{v}) = M_{\theta}(\mathbf{v}) + \mathbf{a} = (M_{\theta}\mathbf{v} + \mathbf{b}) + \mathbf{c}$ . By the perpendicular case,  $M_{\theta}\mathbf{v} + \mathbf{b}$  is a reflection in an axis at an angle  $\theta/2$  to the positive x-axis. Then since  $\mathbf{c}$  is parallel to this axis the composite is a glide in that axis.



## PLANE ISOMETRIES

TYPE	DIR- ECT?	FIXED POINTS	FIXED LINES
<b>IDENT</b>	√	All	all
<b>ROT'N</b>	√	Centre ●	none
<b>TRANS'N</b>	√	None	none
<b>REFL'N</b>	×	● ● ● ● ● ● ● ● all points on a line	 a line and all lines ⊥ to it
<b>GLIDE</b>	×	None	 a line

We denote the isometries of the plane as follows:

**I** = identity isometry;

**R<sub>a,θ</sub>** = rotation about **a** through the angle  $\theta$ ;

**T<sub>a</sub>** = translation by the vector **a**;

**M<sub>a,b</sub>** = reflection in the line joining the points **a** and **b**;

**G<sub>a,b</sub>** = glide in the line joining the points **a** and **b**.

**Theorem 8:** Let **p** be the foot of the perpendicular from **0** to the line joining **a** to **b**. Then

(i)  $T_a(\mathbf{v}) = \mathbf{v} + \mathbf{a}$ .

(ii)  $R_{a,\theta}(\mathbf{v}) = R_\theta(\mathbf{v} - \mathbf{a}) + \mathbf{a}$ .

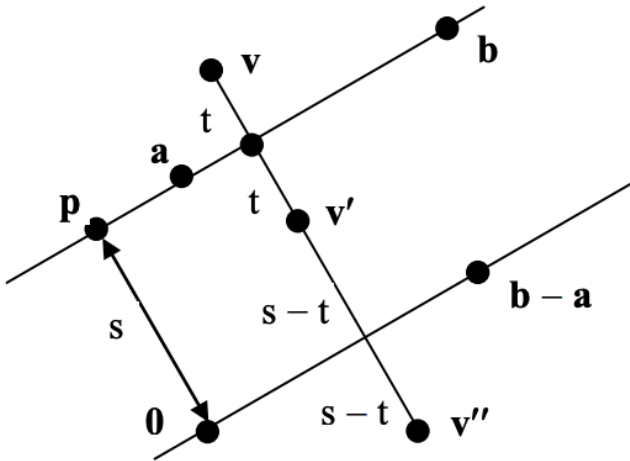
(iii)  $M_{a,b}(\mathbf{v}) = M_{b-a}(\mathbf{v}) + 2\mathbf{p}$

(iv)  $G_{a,b}(\mathbf{v}) = M_{b-a}(\mathbf{v}) + 2\mathbf{p} + \mathbf{b} - \mathbf{a}$ .

**Proof:** (i) follows from the parallelogram law for vector addition:

(ii) can be proved by first translating  $\mathbf{a}$  to the origin, then performing the corresponding central isometry, and then translating back.

(iii) If  $\mathbf{v}'$  is the reflection of  $\mathbf{v}$  in the line joining  $\mathbf{a}$  to  $\mathbf{b}$  and  $\mathbf{v}''$  is the reflection of  $\mathbf{v}$  in the line joining  $\mathbf{0}$  to  $\mathbf{b} - \mathbf{a}$  then  $\mathbf{v}' = \mathbf{v}'' + 2\mathbf{p}$ . Part (iv) is now obvious.

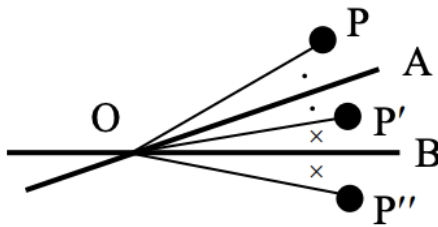


**Theorem 9:** Every plane isometry is a product of at most 3 reflections.

**Proof:**

**Identity:** If  $M$  is any reflection,  $I = M^2$ .

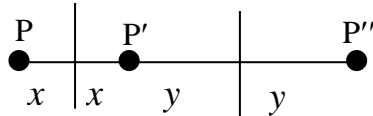
**Rotation:** The product of two reflections whose axes intersect at P is a rotation about P through twice the angle between them.



Reflecting P in OA gives P'. Reflecting this in OB gives P''. The combined effect of these two reflections is to

rotate P through twice the angle  $\angle AOB$ .

**Translation:** The product of two reflections in parallel axes is a translation in the direction perpendicular to the axes, through twice the distance between them.



**Reflection:** A reflection is a ‘product’ of one reflection.

**Glide:** A glide is a reflection followed by a translation, so from the above it is a product of three reflections.

## §6.6. Identifying Products of Plane Isometries

Suppose we're given a sequence of plane isometries, to be performed sequentially, and we wish to identify the product. One way would be to perform the necessary matrix and vector calculations. But then we would want to convert the resulting formula to a geometric description. Now that we know all the possible isometries of the plane we can avoid a lot of the algebra and proceed geometrically.

We know how to multiply products of central isometries easily, but what happens if we have some translations in the product?

**Theorem 10:** If  $f$  is a central isometry then for all  $\mathbf{a}$ ,  $\tau_{\mathbf{a}}f = f\tau_{\mathbf{b}}$  for some  $\mathbf{b}$ .

**Proof:** Suppose  $F$  is the matrix of the central isometry.

$$\begin{aligned}\text{Then } \tau_{\mathbf{a}}f(\mathbf{v}) &= F(\mathbf{v} + \mathbf{a}) \\ &= F\mathbf{v} + F\mathbf{a}.\end{aligned}$$

Then  $\tau_{\mathbf{a}}f = f\tau_{\mathbf{b}}$  for  $\mathbf{b} = F\mathbf{a}$ .

This means that if we have a product of central isometries and translations the product will have the form  $f\tau$  where  $f$  is the product of the central isometries, ignoring the translation, and  $\tau$  will be some translation.

Since we know how to easily multiply central isometries we'll easily be able to determine the factor  $f$  in the answer. All that remains is to determine the  $\tau$ .

**Example 7:** Identify the product of a reflection in the line  $y = x$ , a glide in the  $y$ -axis through a distance of 1 unit vertically and then the rotation through  $90^\circ$  about  $(1, 1)$ .

**Solution:** The reflection is  $\mu_{45}$ , with matrix  $M_{90}$ , the glide is  $\mu_{90}\tau_{(0,1)}$  and the rotation will be  $\rho_{90}\tau$  for some translation  $\tau$ .

The product is  $\mu_{45}\mu_{90}\tau_{(0,1)}\rho_{90}\tau = \mu_{45}\mu_{90}\rho_{90}\tau'$  for some translation  $\tau'$ .

The matrix for  $\mu_{45}\mu_{90}\rho_{90}$  will be  $R_{90}M_{180}M_{90} = R_{90+180-90} = R_{180}$ .

So the product will be  $\rho_{180}\tau'$  and so will be a  $180^\circ$  rotation about some point.

To find the centre of the rotation we choose a point. The midpoint of this point and its image will clearly be the centre of the rotation.

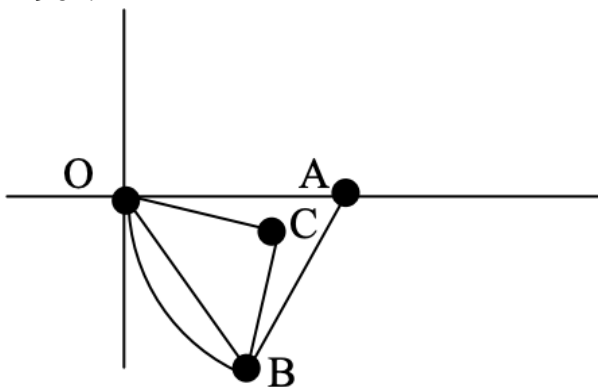
Take  $P = (0, 0)$ . This maps to  $(0, 1)$ . Hence the centre will be  $(0, \frac{1}{2})$ . So the product is the rotation through  $180^\circ$  about  $(0, \frac{1}{2})$ .

**Example 8:** Identify the product of a rotation through  $30^\circ$  about the origin,  $O$ , followed by a rotation through  $60^\circ$  about  $A = (1, 0)$ .

**Solution:** This is clearly a  $90^\circ$  rotation. The only question is: what is the centre?

The origin  $O$  rotates to the point  $B$  where  $AO = AB$  and the angle  $OAB$  is  $60^\circ$ .

The centre must be the point  $C$  such that  $CO = CB$  and  $\angle OCB = 90^\circ$ .



Since  $\triangle OAB$  is an isosceles triangle with a  $60^\circ$  angle it's equilateral, and so  $OB = 1$ .  $\triangle OCB$  is a right-angled triangle and so  $OC = 1/\sqrt{2}$ . Angle  $AOC = 60^\circ - 45^\circ = 15^\circ$ . So  $C$  is the point  $(x, y)$  where:

$$x = \frac{1}{\sqrt{2}} \cos 15^\circ \text{ and } y = -\frac{1}{\sqrt{2}} \sin 15^\circ .$$

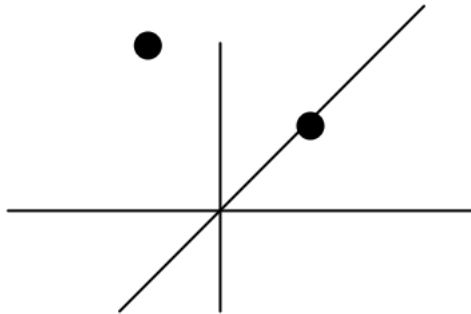
**Example 9:** Identify the product,  $f$ , of a reflection in the line  $y = x$ , the rotation through  $90^\circ$  about  $(1, 1)$  and a glide in the  $y$ -axis through a distance 1 vertically.

**Solution:** Note that these factors are the same as in example 7, but the order is different. The product can be written as  $\mu_{45}\rho_{90}\tau\mu_{90}\tau'$  for some translations  $\tau, \tau'$ . The matrix for  $\mu_{45}\rho_{90}\mu_{90}$  is  $M_{180}R_{90}M_{90}$  (remember to reverse

the order when using the matrices and to double the angle for the  $M$ 's). The product is clearly  $R_{180 - 90 - 90} = R_0$  and so is the identity and so the final answer is a translation (or possibly the identity).

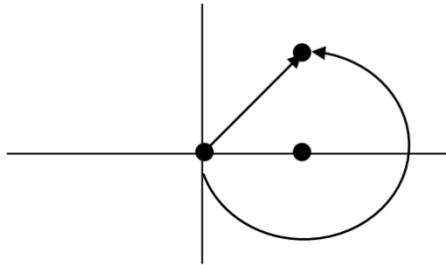
Take  $P = (1, 1)$ . Then  $f(P) = (-1, 2)$ .

Since  $(-1, 2) - (1, 1) = (-2, 1)$  the product is a translation by the vector  $(-2, 1)$ .



**Example 10:** Identify the product of a rotation through  $90^\circ$  about the origin,  $O$ , followed by a rotation through  $270^\circ$  about  $A = (1, 0)$ .

**Solution:** This is clearly  $\rho_{360}\tau = \tau$  for some translation  $\tau$ .

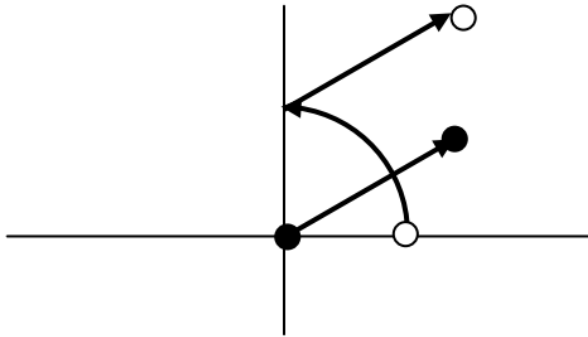


The origin is mapped to  $(1, 1)$  so the product is the translation by the vector  $(1, 1)$ .

**Example 11:** Identify the product of a rotation through  $90^\circ$  about the origin,  $O$ , followed by the translation

$$f(\mathbf{v}) = \mathbf{v} + (1, 1).$$

**Solution:** The product is clearly a rotation through  $90^\circ$ . The only question is: what is the centre? The origin maps to  $(1, 1)$  so the centre  $C$  is such that angle  $OCA$  is  $90^\circ$ .



Clearly  $C$  must be  $(0, 1)$ . ( $C = (1, 0)$  would also make  $OCA$  a right angle, but in that case it would need a rotation through  $270^\circ$  to rotate  $O$  to  $A$ .) So the product is a  $90^\circ$  rotation about  $(0, 1)$ .

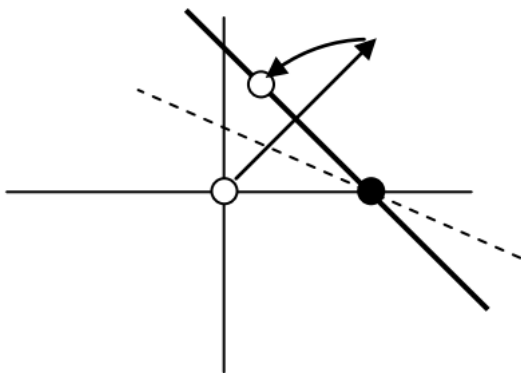
**Example 12:** Identify the product of the reflection in  $y = x + 3$  followed by the reflection in  $x + y = 1$ .

**Solution:** The product is  $\mu_{45}\tau\mu_{-45}\tau'$  for some translations  $\tau, \tau'$ . The matrix of  $\mu_{45}\mu_{-45}$  is  $M_{-90}M_{90} = M_{-180} = M_{180}$ .

So it is a  $180^\circ$  rotation. Clearly it must fix the intersection of the two lines, namely  $(-1, 2)$ . So the product is the rotation through  $180^\circ$  about  $(-1, 2)$ .

**Example 13:** Identify the product of the reflection in the line  $x + y = 1$  followed by the rotation through  $45^\circ$  about the point  $(1, 0)$ .

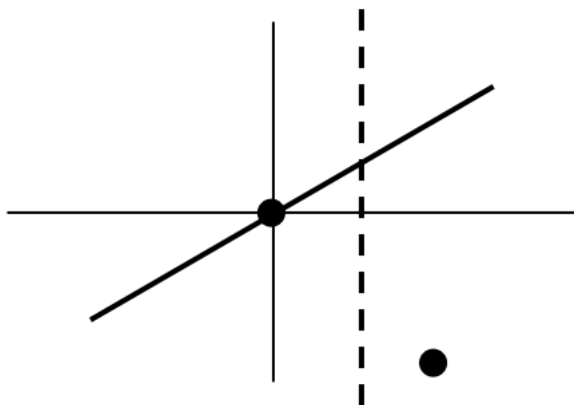
**Solution:**  $R_{45}M_{-90} = M_{-45}$  so the product must be a reflection in a line inclined to the positive x-axis by the angle  $-22.5^\circ$ . Since the point  $(1, 0)$  lies on the axis of reflection it is fixed by the product. Therefore the product must be a reflection in the line through  $(1, 0)$  at an angle of  $-22.5^\circ$ .



**Example 14:** Identify the product of the reflection in the line  $y = x$  followed by the rotation through  $90^\circ$  about the point  $(1, 0)$ .

**Solution:**  $R_{90}M_{90} = M_{180}$  so the product is either a reflection or a glide in a vertical axis. The origin moves

to the point  $(1, -1)$  so the product must be the glide which consists of a reflection in the line  $x = \frac{1}{2}$  followed by a translation vertically downwards through a distance of 1 unit.



## EXERCISES FOR CHAPTER 6

### Exercise 1:

Consider the following isometries of the plane:

T is the translation to the right through a distance of 1 unit;

R is the  $90^\circ$  (anti-clockwise) rotation about  $(1, 1)$ ;

M is the reflection in the line  $x + y = 1$ .

Identify each of the following products as a translation, a rotation, a reflection or a glide. If a product is a translation, or a glide, state the direction and the distance. If it's a rotation, state the centre and the angle. If it's a reflection state the axis.

(a) TR;

(b) MT;

**Exercise 2:** The point  $P(1, 1)$  is rotated through  $30^\circ$  about the point  $(2, 3)$  and then translated in the direction of  $(1, 2)$  through a distance of 3 units. Find the coordinates of the resulting point.

**Exercise 3:** ABCD is a unit square and a point P is successively rotated through  $90^\circ$  about each of the four points, in the given order. Show that, after the four rotations, the net effect will be to translate P in the direction AD through a distance of 4 units.

**Exercise 4:** Let  $O = (0, 0)$ ,  $A = (0, 1)$  and let B be such a point in the upper half plane that makes OAB an equilateral triangle. A point P is rotated through  $30^\circ$  about

O, then through  $30^\circ$  about A and finally through  $30^\circ$  about B. It is clear that the result is a  $90^\circ$  rotation. Find the centre of this rotation.

## SOLUTIONS FOR CHAPTER 6

### Exercise 1:

(a) (i) TR is a direct isometry which rotates all lines by  $90^\circ$  so it must be a  $90^\circ$  rotation about some point C.

Under TR:  $(0, 0) \rightarrow (1, 0) \rightarrow (2, 1)$  so C lies on the perpendicular bisector of the interval from  $(0, 0)$  to  $(2, 1)$ .

The midpoint is  $(1, \frac{1}{2})$  and the slope of the perpendicular is  $-2$ .

Thus C lies on the line  $y = -2x + 5/2$ .

Similarly  $(1, 1) \rightarrow (2, 1) \rightarrow (1, 2)$ .

The midpoint is  $(1, 3/2)$  and the perpendicular bisector is horizontal. So its equation is  $y = 3/2$ .

Solving these two equations we get  $x = 1/2$  and  $y = 3/2$ .

So TR is a  $90^\circ$  rotation about  $(1/2, 3/2)$ .

(b) MT is an opposite isometry.

Under  $(MT)^2$ :  $(0, 0) \rightarrow (1, 1) \rightarrow (1, 1) \rightarrow (0, 0) \rightarrow (2, 0)$  so  $(MT)^2 \neq I$ . Hence MT is not a reflection. Therefore it must therefore be a glide.  $(MT)^2$  must therefore be a translation. It must, in fact, be a translation through 2 units to the right. So MT must be a glide in a horizontal line through a distance of 1 unit. So MR is a glide along the axis  $y = 1/2$  through a distance of 1 unit to the right.

**Exercise 2:**

Translate (2, 3) to the origin. So (1, 1)  $\rightarrow$  (1, -1).

The rotation matrix for a  $30^\circ$  rotation about the origin is

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \text{ so } (1, -1) \rightarrow \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{3}}{2} \\ \frac{\sqrt{3}-1}{2} \end{pmatrix}.$$

Now translate the origin back to (2, 3). The point now moves to  $\left(\frac{3-\sqrt{3}}{2}, \frac{\sqrt{3}+5}{2}\right)$ .

Now translate by  $\frac{3}{\sqrt{2}}(1, 2)$  to get:

$$\left(\frac{3-\sqrt{3}}{2} + \frac{3\sqrt{5}}{5}, \frac{\sqrt{3}+5}{2} + \frac{6\sqrt{5}}{5}\right)$$

**Exercise 3:** Let the four points be given by the vectors  $\mathbf{0}$ ,  $\mathbf{a}$ ,  $\mathbf{a} + \mathbf{b}$  and  $\mathbf{b}$ , with  $\mathbf{a}$ ,  $\mathbf{b}$  being an orthonormal basis of  $\mathbb{R}^2$ . Let  $\mathbf{R} = \mathbf{R}_{90} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  be the matrix of a  $90^\circ$  rotation

about the origin. Then  $\mathbf{R}\mathbf{a} = \mathbf{b}$  and  $\mathbf{R}\mathbf{b} = -\mathbf{a}$ . If  $\mathbf{P}$  is represented by  $\mathbf{v}$  then the successive positions are:

$$\mathbf{v} \rightarrow \mathbf{R}\mathbf{v}$$

$$\rightarrow \mathbf{R}(\mathbf{R}\mathbf{v} - \mathbf{a}) + \mathbf{a} = \mathbf{R}^2\mathbf{v} - \mathbf{R}\mathbf{a} + \mathbf{a} = -\mathbf{v} - \mathbf{b} + \mathbf{a}$$

$$\rightarrow \mathbf{R}[-\mathbf{v} - \mathbf{b} + \mathbf{a} - (\mathbf{a} + \mathbf{b})] + \mathbf{a} + \mathbf{b}$$

$$= -\mathbf{R}\mathbf{v} - 2\mathbf{R}\mathbf{b} + \mathbf{a} + \mathbf{b}$$

$$= -\mathbf{R}\mathbf{v} + 3\mathbf{a} + \mathbf{b}$$

$$\rightarrow \mathbf{R}[-\mathbf{R}\mathbf{v} + 3\mathbf{a} + \mathbf{b} - \mathbf{b}] + \mathbf{b}$$

$$= -\mathbf{R}^2\mathbf{v} + 3\mathbf{R}\mathbf{a} + \mathbf{b} = \mathbf{v} + 4\mathbf{b}.$$

So the product of the four rotations is a translation in the direction AD through a distance of 4.

[It's clear that the product of four  $90^\circ$  rotations about any centres will result in directed lines being fixed and so will be a translation. If we take A and carry out the rotations we can see geometrically that the direction and distance of the translation are as claimed.]

**Exercise 4:** Let A be represented by the vector  $\mathbf{a}$  and let B be represented by  $\mathbf{b}$ .

Let  $c = \cos 30^\circ = \sqrt{3}/2$  and  $s = \sin 30^\circ = 1/2$  and let

$\mathbf{R} = \mathbf{R}_{30} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$  be the matrix of a  $30^\circ$  rotation about the origin. Then  $\mathbf{b} = \mathbf{R}^2\mathbf{a}$ .

If P is represented by the vector  $\mathbf{v}$  then:

$$\mathbf{v} \rightarrow \mathbf{R}\mathbf{v}$$

$$\rightarrow \mathbf{R}(\mathbf{R}\mathbf{v} - \mathbf{a}) + \mathbf{a} = \mathbf{R}^2\mathbf{v} - \mathbf{R}\mathbf{a} + \mathbf{a}$$

$$\rightarrow \mathbf{R}(\mathbf{R}^2\mathbf{v} - \mathbf{R}\mathbf{a} + \mathbf{a} - \mathbf{R}^2\mathbf{a}) + \mathbf{R}^2\mathbf{a} = \mathbf{R}^3\mathbf{v} + \mathbf{R}\mathbf{a} - \mathbf{R}^3\mathbf{a}.$$

Suppose the centre is represented by the vector  $\mathbf{z}$ .

$$\text{Then } \mathbf{v} \rightarrow \mathbf{R}^3(\mathbf{v} - \mathbf{z}) + \mathbf{z} = \mathbf{R}^3\mathbf{v} - \mathbf{R}^3\mathbf{z} + \mathbf{z}.$$

$$\text{Hence } (\mathbf{I} - \mathbf{R}^3)\mathbf{z} = \mathbf{R}\mathbf{a} - \mathbf{R}^3\mathbf{a}$$

$$\text{And so } \mathbf{z} = \mathbf{R}(\mathbf{I} - \mathbf{R})(\mathbf{I} + \mathbf{R} + \mathbf{R}^2)^{-1}\mathbf{a}$$

$$\begin{aligned} \text{Now } \mathbf{I} + \mathbf{R} + \mathbf{R}^2 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} c & -s \\ s & c \end{pmatrix} + \begin{pmatrix} s & -c \\ c & s \end{pmatrix} \\ &= \begin{pmatrix} 1+c+s & -(s+c) \\ (s+c) & 1+c+s \end{pmatrix} = \begin{pmatrix} 1+t & -t \\ t & 1+t \end{pmatrix} \end{aligned}$$

where  $t = c + s$ .

$$(\mathbf{I} + \mathbf{R} + \mathbf{R}^2)^{-1} = \frac{1}{1+2t+2t^2} \begin{pmatrix} 1+t & -t \\ -t & 1+t \end{pmatrix}.$$

$$\begin{aligned} (\mathbf{I} + \mathbf{R} + \mathbf{R}^2)^{-1} \mathbf{a} &= \frac{1}{1+2t+2t^2} \begin{pmatrix} 1+t & -t \\ -t & 1+t \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{1+2t+2t^2} \begin{pmatrix} 1+t \\ -t \end{pmatrix}. \end{aligned}$$

$$\begin{aligned} \mathbf{R} + \mathbf{R}^2 &= \begin{pmatrix} t & -t \\ t & t \end{pmatrix} \text{ so } \mathbf{z} = \frac{1}{1+2t+2t^2} \begin{pmatrix} t-t \\ t & t \end{pmatrix} \begin{pmatrix} 1+t \\ -t \end{pmatrix} \\ &= \frac{1}{1+2t+2t^2} \begin{pmatrix} t+2t^2 \\ t \end{pmatrix} \end{aligned}$$

So the centre is  $\left( \frac{t+2t^2}{1+2t+2t^2}, \frac{t}{1+2t+2t^2} \right)$ .

Now  $t = c + s = \frac{1 + \sqrt{3}}{2}$  and  $t^2 = \frac{2 + \sqrt{3}}{2}$ .

$$1+2t+2t^2 = 4 + 2\sqrt{3} \text{ and } t + 2t^2 = \frac{5+3\sqrt{3}}{2}.$$

So the centre of the equivalent  $90^\circ$  rotation is  $\left( \frac{5+3\sqrt{3}}{8+4\sqrt{3}}, \frac{1+\sqrt{3}}{8+4\sqrt{3}} \right) = \left( \frac{1+\sqrt{3}}{28}, \frac{-1+\sqrt{3}}{28} \right)$ .

